Stiffness matrices have also been calculated by taking terms up to β_7, β_9, and β_{10}, respectively. It turns out that, when only five significant figures are kept, the three resulting matrices are identical, indicating that the convergency of the solution is indeed very rapid. This resulting matrix is given in the following:

$$
Et \begin{bmatrix}
0.46875 & -0.28125 & -0.28125 & 0.09375 & 0.18750 & 0.00000 & -0.18750 & 0.00000 \\
 & 0.46875 & 0.09375 & -0.28125 & 0.00000 & -0.18750 & 0.00000 & 0.18750 \\
 & & 0.46875 & -0.28125 & -0.18750 & 0.00000 & 0.18750 & 0.00000 \\
 & & & 0.46875 & 0.00000 & 0.18750 & 0.00000 & -0.18750 \\
 & & & & 0.46875 & 0.09375 & -0.28125 & -0.28125 \\
 & \text{symmetric} & & & & 0.46875 & -0.28125 & -0.28125 \\
 & & & & & & 0.46875 & 0.09375 \\
 & & & & & & & 0.46875
\end{bmatrix}
$$

Finally, it should be mentioned that the foregoing converging value of the element stiffness matrix can also be obtained by employing the principle of minimum potential energy[1] if a sufficiently large number of terms of the following displacement functions are used:

$$
\begin{aligned}
u &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + xy(x-a)(y-b)(\alpha_9 + \alpha_{11}x + \alpha_{13}y + \ldots) \\
v &= \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy + xy(x-a)(y-b)(\alpha_{10} + \alpha_{12}y + \alpha_{14}x + \ldots)
\end{aligned} \tag{25}
$$

For example, when terms up to α_8 are used, the resulting square element stiffness matrix by Eq. (15) of Ref. 1 is as follows:

$$
Et \begin{bmatrix}
0.5 & -0.3125 & -0.25 & 0.0625 & 0.1875 & 0 & -0.1875 & 0 \\
 & 0.5 & 0.0625 & -0.25 & 0 & -0.1875 & 0 & 0.1875 \\
 & & 0.5 & -0.3125 & -0.1875 & 0 & 0.1875 & 0 \\
 & & & 0.5 & 0 & 0.1875 & 0 & -0.1875 \\
 & & & & 0.5 & 0.0625 & -0.25 & -0.3125 \\
 & \text{symmetric} & & & & 0.5 & -0.3125 & -0.25 \\
 & & & & & & 0.5 & 0.0625 \\
 & & & & & & & 0.5
\end{bmatrix}
$$

When terms up to α_{10} are used, the resulting matrix is

$$
Et \begin{bmatrix}
0.47396 & -0.28646 & -0.27604 & 0.08854 & 0.1857 & 0 & -0.1875 & 0 \\
 & 0.47396 & 0.08854 & -0.27604 & 0 & -0.1875 & 0 & 0.1875 \\
 & & 0.47396 & -0.28646 & -0.1875 & 0 & 0.1875 & 0 \\
 & & & 0.47396 & 0 & 0.1875 & 0 & -0.1875 \\
 & & & & 0.47396 & 0.08854 & -0.27604 & -0.28646 \\
 & \text{symmetric} & & & & 0.47396 & -0.28646 & -0.27604 \\
 & & & & & & 0.47396 & 0.08854 \\
 & & & & & & & 0.47396
\end{bmatrix}
$$

It is seen that the result is converging, although not as rapidly as that by the method of assumed stress functions.

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A Simple Derivation of Three-Dimensional Characteristic Relations

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The method of characteristics for three-dimensional flow has been studied by a number of investigators.[1–5] Most of the existing derivations of the conditions for characteristic surfaces and the compatibility relations seem either general and lengthy or particular and sketchy. This note presents a simple, straightforward derivation of the characteristic surfaces and compatibility relations of steady, inviscid, isoenergetic, three-dimensional flow, based on a basic idea of characteristics.

The fundamental equations in the present case are

$$
(u^2 - a^2)u_x + (v^2 - a^2)v_y + (w^2 - a^2)w_z + uv(v_x + u_y) + vw(w_y + v_z) + wu(u_z + w_x) = 0 \tag{1}
$$

$$
v(v_x - u_y) - w(u_z - w_x) + a^2 s_x' = 0 \tag{2}
$$

$$
w(w_y - v_z) - u(v_x - u_y) + a^2 s_y' = 0 \tag{3}
$$

$$
u(u_z - w_x) - v(w_y - v_z) + a^2 s_z' = 0 \tag{4}
$$

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where u, v, or w is the x, y, or z component of the velocity vector; a is the speed of sound; $\gamma R s' = s$ is the entropy per unit mass; and subscript x, y, or z denotes partial differentiation with respect to x, y, or z.

As is well known, a characteristic surface allows possible discontinuity of derivatives of flow quantities in its normal direction; in other words, given initial data on such a surface, the normal derivatives of flow quantities are not uniquely determined by the fundamental equations (1–4). Now, let the rectangular coordinates (x, y, z) be so oriented that the x and y axes are tangent to a characteristic surface at the origin, which may be located anywhere in the flow field; then the normal derivative w_z is not uniquely determined by Eqs. (1–4). Solving (1–3)† for w_z in terms of u, v, w, s', and their derivatives with respect to x and y, we obtain $w_z = detN/detD$, where

$$detN = \begin{vmatrix} (a^2 - u^2)u_x + (a^2 - v^2)v_y - & & \\ \quad uv(v_x + u_y) - w(uw_x + vw_y) & u & v \\ v(u_y - v_x) - ww_x - a^2 s_x' & -1 & 0 \\ u(v_x - u_y) - ww_y - a^2 s_y' & 0 & -1 \end{vmatrix}$$

and

$$detD = \begin{vmatrix} w^2 - a^2 & u & v \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

A necessary and sufficient condition for w_z to be indeterminate is $detD = 0$ and $detN = 0$. The first equation leads to $w^2 - a^2 = 0$, or

$$w = \pm a \tag{5}$$

The second equation yields

$$(w^2 - u^2)u_x + (w^2 - v^2)v_y - uv(v_x + u_y) - 2w(uw_x + vw_y) - a^2(us_x' + vs_y') = 0 \tag{6}$$

Equation (5) states that the velocity component normal to a characteristic surface is equal to the local speed of sound; in other words, a characteristic surface is everywhere tangent to the local Mach cone. Substitution of (5) into (6) yields the corresponding compatibility relation

$$(a^2 - u^2)u_x + (a^2 - v^2)v_y - uv(v_x + u_y) \mp 2a(uw_x + vw_y) - a^2(us_x' + vs_y') = 0 \tag{7}$$

Similarly, solving (2–4) for s_z', we find

$$s_z' = -(us_x' + vs_y')/w$$

The condition that s_z' is indeterminate leads to $w = 0$, $us_x' + vs_y' = 0$. Hence, we see that all surfaces composed of streamlines are also characteristic surfaces and that the corresponding compatibility relation is the constancy of entropy along streamlines. It may be noted that stream surfaces are no longer characteristic when, in addition, the flow is assumed to be irrotational,[3] since s_z' is now identically zero.

When the y axis is chosen to coincide with a generatrix of the Mach cone, the velocity component u vanishes and Eq. (7) assumes the simpler form

$$(a^2 - v^2)v_y + a^2 u_x \mp 2avw_y - a^2 v s_y' = 0 \qquad \text{or}$$

$$(1 - \cot^2\mu)v_y + u_x \mp 2\cot\mu w_y - (a^2 \cot\mu s_y'/q \sin\mu) = 0 \tag{7'}$$

where μ is the Mach angle. It may be noted that (7′) can be transformed into Ferri's form[1] in the velocity-oriented coordinates (see Appendix).

In conclusion, it is noted that the present approach may be applied to general unsteady three-dimensional flow and, in fact, to hyperbolic equations in n dimensions.

† The same result will be obtained by including (4), which is redundant for finding w_z.

Appendix

We will show that (7′) can be transformed into the forms of other investigators. The rectangular coordinates will first be rotated by an angle $\pm\mu$ with the x axis kept fixed so that the new y' axis coincides with the velocity vector. Let v' and w' denote the velocity components in the new y' and z' direction; the following transformation relations are valid:

$$v_y = \cos^2\mu v_{y'}' \mp \sin\mu \cos\mu(v_{z'}' + w_{y'}') + \sin^2\mu w_{z'}'$$

$$w_y = \cos^2\mu w_{y'}' \mp \sin\mu \cos\mu(w_{z'}' - v_{y'}') - \sin^2\mu v_{z'}'$$

$$s_y' = \cos\mu s_{y'}' \mp \sin\mu s_{z'}' = \mp \sin\mu s_{z'}'$$

since $s_{y'}' = 0$. Substitution of these relations into (7′) yields

$$-\cot\mu v_{y'}' \pm v_{z'}' \mp w_{y'}' + \tan\mu(w_{z'}' + u_x) \pm (a^2/q)s_{z'}' = 0$$

or, with M denoting the Mach number,

$$-(M^2 - 1)v_{y'}' \pm (M^2 - 1)^{1/2}(v_{z'}' - w_{y'}') + w_{z'}' + u_x \pm (a^2/q)(M^2 - 1)^{1/2}s_{z'}' = 0 \tag{8}$$

which is a compatibility relation that can be obtained by straightforward application of the general theory of Ref. 3. Now, Eq. (8) can be reduced to (20-19) of Ref. 1 when it is noted that y', z', and x correspond to t, n, and N, and when Eqs. (20-11, 20-15, and 20-16) of Ref. 1 are used.

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Extension of *f* and *g* Series to Non-Two-Body Forces

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Background

BASIC to a number of our orbit determination on prediction routines is the use of the classical f and g series of celestial mechanics. This series, in turn, is based upon an assumption of two-body motion.

In many applications of the f and g series expressions, the accuracy to which the f_i and g_i are developed is far in excess of the accuracy with which the two-body problem truly represents the physical orbit. Such an occurrence poses no difficulty when the f and g series is used for the generation of a reference two-body orbit, but it does pose a problem if it is used indiscriminately to represent observations accurately or to underlie a definitive ephemeris.

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